## Inequalities

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## 1 Arithmetic Mean-Geometric Mean Inequality

Theorem 1. AM-GM states that for any set of nonnegative real numbers, the arithmetic mean of the set is greater than or equal to the geometric mean of the set. Algebraically, this is expressed as follows.
For a set of nonnegative real numbers $a_{1}, a_{2}, \ldots, a_{n}$ the following always holds:

$$
\frac{a_{1}+a_{2}+\ldots+a_{n}}{n} \geq \sqrt[n]{a_{1} a_{2} \ldots . a_{n}}
$$

The equality condition of this inequality states that the arithmetic mean and geometric mean are equal if and only if all members of the set are equal.

1. Find the maximum of $2-a-\frac{1}{2 a}$ for all positive $a$.
2. Let $a, b, c$ be positive real numbers such that $a^{2}+b^{2}+c^{2}=1$. Show that

$$
a(b+c) \leq \frac{\sqrt{2}}{2}
$$

We also could have solved it by applying AM-GM inequality
3. Prove that for positive real numbers the following inequality holds

$$
(a+b-c)(b+c-a)(a+c-b) \leq a b c
$$

4. (2011 USAMO Problems/Problem 1) Let $a, b, c$ be positive real numbers such that $a^{2}+$ $b^{2}+c^{2}+(a+b+c)^{2} \leq 4$. Prove that

$$
\frac{a b+1}{(a+b)^{2}}+\frac{b c+1}{(b+c)^{2}}+\frac{a c+1}{(a+c)^{2}} \geq 3
$$

## 2 CAUCHY-SCHWARZ

The Cauchy-Schwarz Inequality (which is known by other names, including Cauchy's Inequality, Schwarz's Inequality, and the Cauchy-Bunyakovsky-Schwarz Inequality) is a well-known inequality with many elegant applications. It has an elementary form, a complex form, and a general form.

Louis Cauchy wrote the first paper about the elementary form in 1821. The general form was discovered by Bunyakovsky in 1849 and independently by Schwarz in 1888.
(Cauchy-Schwarz Inequality) For any real numbers $a_{1}, \ldots, a_{n}$ and $b_{1}, \ldots, b_{n}$,

$$
\left(\sum_{i=1}^{n} a_{i} b_{i}\right)^{2} \leq\left(\sum_{i=1}^{n} a_{i}^{2}\right)\left(\sum_{i=1}^{n} b_{i}^{2}\right)
$$

with equality when there exists a nonzero constant $\mu$ such that for all $1 \leq i \leq n, \mu a_{i}=b_{i}$.

## Discussion:

Consider the vectors $\mathbf{a}=\left\langle a_{1}, \ldots a_{n}\right\rangle$ and $\mathbf{b}=\left\langle b_{1}, \ldots b_{n}\right\rangle$. If $\theta$ is the angle formed by $\mathbf{a}$ and $\mathbf{b}$, then the left-hand side of the inequality is equal to the square of the dot product of $\mathbf{a}$ and $\mathbf{b}$, or $(\mathbf{a} \cdot \mathbf{b})^{2}=a^{2} b^{2}(\cos \theta)^{2}$. The right hand side of the inequality is equal to $(\|\mathbf{a}\| *\|\mathbf{b}\|)^{2}=a^{2} b^{2}$. The inequality then follows from $|\cos \theta| \leq 1$, with equality when one of $\mathbf{a}, \mathbf{b}$ is a multiple of the other, as desired.
5. Let $a+b+c=1$. Prove that

$$
a^{2}+b^{2}+c^{2} \geq \frac{1}{3}
$$

6. (2009 USAMO Problems/Problem 4) For $n \geq 2$ let $a_{1}, a_{2}, \ldots, a_{n}$ be positive real numbers such that $\left(a_{1}+a_{2}+\ldots+a_{n}\right)\left(\frac{1}{a_{1}}+\frac{1}{a_{2}}+\ldots+\frac{1}{a_{n}}\right) \leq\left(n+\frac{1}{2}\right)^{2}$.
Prove that $\max \left(a_{1}, a_{2}, \ldots, a_{n}\right) \leq 4 \min \left(a_{1}, a_{2}, \ldots, a_{n}\right)$.
7. (2004 USAMO Problems/Problem 5) Let $a, b, c$ be positive real numbers. Prove that

$$
\left(a^{5}-a^{2}+3\right)\left(b^{5}-b^{2}+3\right)\left(c^{5}-c^{2}+3\right) \geq(a+b+c)^{3}
$$

## 3 Jensen’s Inequality

Theorem 2. Let $F$ be a convex function of one real variable. Let $x_{1}, \ldots, x_{n} \in \mathbb{R}$ and let $a_{1}, \ldots, a_{n} \geq 0$ satisfy $a_{1}+\cdots+a_{n}=1$. Then

$$
F\left(a_{1} x_{1}+\cdots+a_{n} x_{n}\right) \leq a_{1} F\left(x_{1}\right)+\cdots+a_{n} F\left(x_{n}\right)
$$

If $F$ is a concave function, we have:

$$
F\left(a_{1} x_{1}+\cdots+a_{n} x_{n}\right) \geq a_{1} F\left(x_{1}\right)+\cdots+a_{n} F\left(x_{n}\right)
$$

8. Let $\alpha, \beta$ and $\gamma$ be the angles in a triangle. Show that

$$
\frac{1}{\sin \left(\frac{\alpha}{2}\right)}+\frac{1}{\sin \left(\frac{\beta}{2}\right)}+\frac{1}{\sin \left(\frac{\gamma}{2}\right)} \geq 6
$$

9. (2001 IMO Problems/Problem 2) Let $a, b, c$ be positive real numbers. Prove that

$$
\frac{a}{\sqrt{a^{2}+8 b c}}+\frac{b}{\sqrt{b^{2}+8 c a}}+\frac{c}{\sqrt{c^{2}+8 a b}} \geq 1
$$

## Homework

1. Write solutions of the problems we did.
2. Solve the rest of the problems and try different ways for practice.
